An Introduction to Black Hole Evaporation

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ABSTRACT

Classical black holes are defined by the property that things can go in, but don't come out. However, Stephen Hawking calculated that black holes actually radiate quantum mechanical particles. The two important ingredients that result in back hole evaporation are (1) the spacetime geometry, in particular the black hole horizon, and (2) the fact that the notion of a "particle" is not an invariant concept in quantum field theory. These notes contain a step-by-step presentation of Hawking's calculation. We review portions of quantum field theory in curved spacetime and basic results about static black hole geometries, so that the discussion is self-contained. Calculations are presented for quantum particle production for an accelerated observer in flat spacetime, a black hole which forms from gravitational collapse, an eternal Schwarzschild black hole, and charged black holes in asymptotically deSitter spacetimes. The presentation highlights the similarities in all these calculations. Hawking radiation from black holes also points to a profound connection between black hole dynamics and classical thermodynamics. A theory of quantum gravity must predicting and explain black hole thermodynamics. We briefly discuss these issues and point out a connection between black hole evaportaion and the positive mass theorems in general relativity.

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1 Introduction

Stephen Hawking published his paper "Particle Creation by Black Holes" [1] in 1975. In this article, Hawking demonstrated that classical black holes radiate a thermal flux of quantum particles, and hence can be expected to evaporate away. This result was contrary to everything that was known about black holes and classical matter, and was quite startling to the physics community. However, the effect has now been computed in a number of ways and is considered an important clue in the search for a theory of quantum gravity. Any theory of quantum gravity that is proposed must predict black hole evaporation. The aim of these notes is to (1) develop enough of the formalism of semi-classical gravity to be able to understand the preceeding sentences, excepting the term "quantum gravity" itself, and (2) give a step-by-step presentation of Hawking's calculation. We will also present a number of related results on particle production for an accelerating observer in flat spacetime, and for charged black holes in asymptotically deSitter spacetimes. Finally, we will discuss an interesting relationship between classical positive mass theorems in general relativity and endpoints of the quantum mechanical process of Hawking evaporation.

For the record, Einstein's equation is given by

$$G_{ab} \equiv R_{ab} - \frac{1}{2} g_{ab} R = 8\pi G_N T_{ab}. \tag{1.1}$$

Here G_{ab} is the Einstein tensor, R_{ab} the Ricci tensor, $R = R^a{}_a$ is the scalar curvature, T_{ab} is the stress-energy tensor and G_N is Newton's gravitational constant. The other constants of nature that come into the calculations are the speed of light c and Planck's constant $\hbar$. In most of the paper, we will work in units with $G = c = \hbar = 1$.

A black hole is a region in an asymptotically flat spacetime which is not contained in the past of future null infinity $\mathcal{I}^+$. The horizon is the boundary between the black hole and the outside, asymptotically flat region. In section (6) we will study a black hole in a spacetime which is not asymptotically flat using an obvious generalization of the definition. The horizon is a null surface. Physically, it is the outer boundary of the black hole on which null rays can just skim along, neither being captured by the black hole, nor propagating to null infinity.

Classical black hole mechanics can be summarized in following three basic theorems, where the necessary symbols are defined in section (4) below.

0) The zeroth law states that the surface gravity κ of a black hole is constant on the horizon.
1) The first law states that variations in the mass M, area A, angular momentum L, and charge Q of a black hole obey [3, 4]

$$\delta M = \frac{\kappa}{8\pi}\delta A + \Omega \delta L - \nu \delta Q, \tag{1.2}$$

where Ω is the angular velocity of the horizon and ν is the difference in the electrostatic potential between infinity and the horizon.
2) The second law is the area theorem [2] proved by Hawking in 1971. The area of a black hole horizon is nondecreasing in time,

$$\delta A \geq 0 \tag{1.3}$$

This result assumes that the spacetime is globally hyperbolic and that the energy condition $R_{ab}k^a k^b \geq 0$ holds for all null vectors k^a.

These theorems bear a striking resemblance to the correspondingly numbered laws of classical thermodynamics. The zeroth law of thermodynamics says that the temperature T is constant throughout a system in thermal equilibrium. The first law states that in small variations between equilibrium configurations of a system, the changes in the energy M and entropy S of the system obey equation 1.2, if $\frac{\kappa}{8\pi}\delta A$ is replaced by $T\delta S$, and the further terms on the right hand side are interpreted as work terms. The second law of thermodynamics states that, for a closed system, entropy always increases in any process, $\delta S \geq 0$.

We see that the theorems describing black hole interactions, which are results from differential geometry, are formally identical to the laws of classical thermodynamics, if one identifies the black hole surface gravity κ with a multiple of T and the area of the horizon A with a multiple of the entropy S. It is tempting to wonder whether this identification is more than formal. Such a conjecture seems to require a drastic shift in the meaning of the geometrical properties of a black hole. Temperature is a measure of the mean energy of a system with a large number, *e.g.* order 10^{23}, of degrees of freedom. Entropy measures the number of microscopic ways these degrees of freedom can be arranged to give a fixed macroscopic configuration, *e.g* fixed M, L and Q. It is not at all obvious that the surface gravity and area of a black hole should have anything to do with a statistical system with a large number of degrees of freedom. Even more glaring, is the problem of radiation. A hot lump of coal radiates. And the definition of a black hole is that it does not radiate; things go in, but don't come out.

Nonetheless, in 1973 Bekenstein [10] suggested that a physical identification does hold between the laws of thermodynamics and the laws of black hole mechanics. Then in 1975, Hawking published his calculation that black holes do indeed radiate, if one takes into account the quantum mechanical nature of matter fields in the spacetime.

2 Quantum Fields in Curved Spacetimes

The Basic Idea of Particle Production

The basic idea of semiclassical gravity is that, for energies below the Planck scale, it is a good approximation to treat matter fields quantum mechanically, but keep gravity classical. Hence, one considers quantum field theory in a fixed curved background. We will focus on free scalar field that classically satisfies the wave equation

$$g^{ab}\nabla_a\nabla_b\phi = 0 \tag{2.4}$$

The scalar field ϕ is a quantum operator. This means that (1) ϕ must obey the canonical equal time commutation relations $[\phi(t,x^i),\phi(t,y^i)] = \delta^3(x^i - y^i)$, and (2) we must define a Hilbert space of states on which these operators act. Physical observables are then computed by taking expectation values of the corresponding operators in a given state, or more generally matrix elements between states.

The key idea behind quantum particle production in curved spacetime is that the definition of a particle is observer dependent. It depends on the choice of reference frame. For example, an observer Al has a natural time coordinate defined by proper time T along Al's world line. As we will discuss in more detail below, Al defines particles as positive frequency oscillations of the scalar field with repect to this time T. A second observer, Emily, will define particles as positive frequency oscillations with respect to her own proper time t. In general, the number of T-particles that Al measures will be different than the number of t-particles that Emily measures. This effect occurs even in flat spacetime [21, 20]. Since quantum field theory in flat spacetime is globally Lorentz invariant, if Al and Emily's frames differ only by a Lorentz transformation, then they *will* agree about particle content. However, if they have a relative acceleration, then they will measure different particle numbers. In the next section, we will study the case when Al uses global inertial coordinates, while Emily undergoes constant acceleration. We will see that in this case, when Al measures spacetime to be empty of his T-particles, Emily will measure this same state to contain a thermal flux of her t-particles.

In general relativity there are more possibilities. Since the theory is generally covariant, any time coordinate, possibly defined only locally within a patch, is a legitimate choice with which to define particles. Of course in a given spacetime, there may be particular choices for coordinates that are more interesting than others from the point of view of physical interpretation. For example, far from a star spacetime becomes flat, and asymptotically inertial Minkowski coordinates (t, x^i) are useful. Suppose now that the star collapses to form a black hole. Far from the black hole, spacetime is still asymptotically flat. Consider a wave packet which starts far from the star and propagates through the collapsing star, such that it just escapes being captures by the forming black hole and propagates back out to the flat region. Suppose that the wave starts out composed only of positive frequency waves with respect to the time coordinate in the asymptotic region t. When the packet passes just outside of the forming horizon, it is in a high-curvature region. The field evolves so that when it is again far from the black hole, it will be a mixture of positive and negative frequency components. The new, negative frequency part corresponds to quantum-particle production. This is the effect that Hawking calculated in his 1975 paper [1].

Canonical Quantization, Hilbert Space and Particle Number Operators

Next we sketch the mathematical structure necessary for turning the scenario described above into a calculation. A reader who does not know quantum field theory will certainly not be able to master it from the next few paragraphs. However, we have tried to provide a complete enough set of definitions and relations, so that these notes are more or less self-contained. We will be thinking of quantum field theory as a linear algebra system and will ignore the problems of regulating and renormalizing the theory to deal with infinities. Quantum operators will be assumed to be "normal ordered", so that their matrix elements are finite. Complete treatments of quantum field theory in curved spacetime can be found in [18, 7].

One standard way to implement canonical quantization is the following. Choose a complete basis f_ω of solutions to the scalar wave equation (2.4), in the spacetime with metric g_{ab}. As a consequence of the wave equation, the basis functions are orthonormal $(f_\omega, f_{\omega'}) = \delta(\omega - \omega')$ with respect to the conserved inner product

$$(f, h) = -i \int d^3x \sqrt{-g} \left(f \dot{h}^* - \dot{f} h^* \right), \tag{2.5}$$

where the integral is taken over a Cauchy surface and dot denotes a time derivative. For example, in Minkowski spacetime with metric $g_{ab} = \eta_{ab}$, the standard choice of basis functions for a scalar field is the set $\{f_\omega, f_\omega^*\}$, where

$$f_\omega = \frac{1}{\sqrt{2\omega}} e^{-i(\omega t - \vec{k}\cdot\vec{x})} \tag{2.6}$$

and $\omega = +\sqrt{\vec{k}\cdot\vec{k}}$. The modes f_ω are the positive frequency modes.

The quantum field ϕ can be expanded in this basis as

$$\phi = \int d\omega (a_\omega f_\omega + a_\omega^\dagger f_\omega^*), \tag{2.7}$$

where the expansion coefficients a_ω and $a_\omega^\dagger$ are operators. For compactness, we are explicitly writing only the energy eigenvalue ω and suppressing other eigenvalue indices. The canonical commutation relations for the scalar field then imply commutation relations for the mode operators $a_\omega, a^\dagger_\omega$,

$$[a_{\omega'}, a^\dagger_\omega] = \delta(\omega' - \omega), \qquad [a_\omega, a_{\omega'}] = [a^\dagger_\omega, a^\dagger_{\omega'}] = 0. \tag{2.8}$$

The vacuum, or lowest energy state, which we denote $|0>_{in}$, is the state which is annihilated by all the annhilation operators a_ω,

$$a_\omega|0>_{in} = 0 \tag{2.9}$$

for all $\omega > 0$. The standard Fock space of states is then constructed by applying arbitrary products of creation operators to $|0>_{in}$. For example, the state $(a_\omega)^n|0>_{in}$ contains n *in*-particles of energy ω. This is made precise by defining the number operator

$$N^{in}_\omega = a^\dagger_\omega a_\omega, \tag{2.10}$$

so that $< 0|a^{\dagger}_{\omega}{}^{n}(N^{in}_\omega){a_\omega}^n|0> = n$. We are calling these "*in*" particles to agree with later notation.

Let us now introduce a second basis of solutions to the scalar wave equation (2.4) $\{p_\omega, p^*_\omega\}$. The scalar field ϕ has an expansion in this basis as well,

$$\phi = \int d\omega(b_\omega p_\omega + b^\dagger_\omega p^*_\omega), \tag{2.11}$$

with new creation and annhilation operators satisfing the commutation relations

$$[b_{\omega'}, b^\dagger_\omega] = \delta(\omega' - \omega), \qquad [b_\omega, b_{\omega'}] = [b^\dagger_\omega, b^\dagger_{\omega'}] = 0. \tag{2.12}$$

The annhilation operators b_ω define a second vacuum state, $|0>_{out}$, satisfying

$$b_\omega|0>_{out} = 0 \tag{2.13}$$

for all $\omega > 0$. A second Fock space of states is built from $|0>_{out}$ by applying the creation operators $b^\dagger_\omega$. The *out*-particle number operator, N^{out}_ω, measures the number of *out*-particles in a state,

$$N^{out}_\omega = b^\dagger_\omega b_\omega, \tag{2.14}$$

so that, *e.g.* $< 0|b^{\dagger}_{\omega}{}^{n}(N^{out}_\omega){b_\omega}^n|0>_{out} = n$.

Bogoliubov Transformations

In order to calculate particle production, we will need to express the number operator N_ω^{out} for the *out*-particles in terms of the creation and annhilation operators for the *in*-particles. Define the linear transformations which relate one basis to the other by

$$p_\omega \quad = \int d\omega'(\alpha_{\omega\omega'} f_{\omega'} + \beta_{\omega\omega'} f^*_{\omega'}) \tag{2.15}$$

$$f_\omega \quad \int d\omega'(\alpha^*_{\omega'\omega} p_{\omega'} - \beta_{\omega'\omega} p^*_{\omega'}). \tag{2.16}$$

The coefficients in these expansions, $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$, called the Bogolubov coefficents, are given by the inner products

$$\alpha_{\omega\omega'} = (p_\omega, f_{\omega'}), \qquad \beta_{\omega\omega'} = -(p_\omega, f^*_{\omega'}) \tag{2.17}$$

As a consequence of orthonormality of the basis functions, the Bogolubov coefficients satisfy

$$\int d\omega'(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = \delta(\omega - \omega') \tag{2.18}$$

Further, we have the relation between the *out* and *in* mode operators

$$b_\omega = \int d\omega' \left(\alpha^*_{\omega\omega'} a_{\omega'} - \beta^*_{\omega\omega'} a^\dagger_{\omega'}\right). \tag{2.19}$$

We can now evaluate the expression (2.14) for N_ω^{out} in the *in*-vacuum state, with the result

$$_{in} < 0|(N_\omega^{out})|0 >_{in} \equiv {}_{in} < 0|b^\dagger_\omega b_\omega|0 >_{in} = \int d\omega' |\beta_{\omega\omega'}|^2 \tag{2.20}$$

We see that although the *in*-vacuum is empty of *in*-particles, in general it will contain *out*-particles, because these particle states are defined with respect to different time coordinates.

To summarize, for a particular calculation one must specify the state of the system, here taken to be the *in*-vacuum. States and operators may be expanded in terms of different bases for the Hilbert space. In general, a different choice of basis includes a different choice of a time coordinate, and hence a different definition of a particle. We work in the Heisenberg representation in which, once specified, the state of the system is fixed and the operators evolve in time. The expectation values of operators/observables of the quantum field ϕ are computed in the state of the system that has been specified.

In the following we will study three examples of particle production calculations. In each case the strategy will be the same. We will make a choice for the state of the system, and compute the particle content for various observers with their various definitions of particles. These choices are the physics input and are determined by what questions one wants to answer!

3 Accelerating Observers in Flat Spacetime

Consider an observer in flat, Minkowski spacetime who undergoes constant acceleration, *i.e.* the magnitude of his four-acceleration is a constant. We call this observer a Rindler observer. The Rindler observer uses proper time along his worldline as a time coordinate. In this example, we will compute the particle production which he observes, and find an interesting result. The Minkowski vacuum, which is empty of particles defined with respect to a global inertial time coordinate, is populated by a thermal bath of particles according to particle-detectors carried by the accelerating Rindler observer! This example is particularly instructive, because the calculations can be done exactly (there is no scattering), and so one clearly sees how the change of basis works. This calculation is in many standard texts, see *e.g.* [7] for a detailed pedagogical treatment. Our presentation will make use of a different choice of basis functions than those usually empoyed, which will generalize more easily to black hole spacetimes.

For notational simplicity we will work in 1+1 dimensional Minkowski spacetime,

$$ds^2 = -dt^2 + dx^2 = -d\bar{u}d\bar{v} \tag{3.21}$$

where $\bar{u} = t - x, \bar{v} = t + x$ are respectively ingoing and outgoing null coordinates. The 4-dimensional calculation is essentially the same. The standard quantum field theory choice for the positive frequency modes of a massless field are the functions $\psi(x,t) \sim e^{-i\omega(t \pm x)}$. Rindler spacetime is the wedge region I of Minkowski spacetime, shown in figure (2), that is covered by the coordinate patch

$$ds^2 = e^{2a\xi}(-dT^2 + d\xi^2) = -e^{a(v-u)}dudv \tag{3.22}$$

where $u = T - \xi, v = T + \xi$. The Rindler metric (3.22) is just a coordinate transformation of (3.21), with

$$v = \frac{1}{a}ln\bar{v}, \qquad u = -\frac{1}{a}ln(-\bar{u}). \tag{3.23}$$

A Rindler observer at constant spatial coordinate ξ undergoes constant acceleration with magnitude $ae^{-a\xi}$, and the observer's proper time coincides with the coordinate T. A Rindler observer always stays within region I and the boundaries of this wedge, along the lines $(t = \pm x)$, are Cauchy horizons for these observers. The T-translation Killing vector $\frac{\partial}{\partial T}$ has zero norm on these horizons. This corresponds to the fact that $\frac{\partial}{\partial T}$ is a boost Killing vector with respect ot the original Minkowski coordinates in (3.21). Due to the Cauchy horizons, the particle production calculations in Rindler and black holes spacetimes are very similiar.

The conformal (or Penrose) diagrams of 3+1 Minkowski, and 1+1 Minkowski with the Rindler wedge are shown in figures (1) and (2). In general, such diagrams are constructed by conformally compactifing the spacetime. The convention is that null paths are 45 degree lines, so the causal structure can be easily read off. See *e.g.* [8, 6] for details.

Inertial and Rindler Bases

To highlight the similiarities, we will call the Rindler horizons $\mathcal{H}^-$ and $\mathcal{H}^+$, in analogy with an eternal black hole. First, let's define the relevant positive and negative frequency modes, and then turn to the issue of normalization. One basis for the space of solutions to the scalar wave equation (2.4) in the global inertial coordinates (3.21) are the functions f_ω, ${f_\omega}^*$, j_ω and ${j_\omega}^*$ given by

$$f_\omega = \frac{1}{\sqrt{2\pi\omega}}e^{-i\omega\bar{v}}, \qquad j_\omega = \frac{1}{\sqrt{2\pi\omega}}e^{-i\omega\bar{u}}. \tag{3.24}$$

The functions f_ω are positive frequency inward propagating modes, while the functions j_ω give positive frequency outward propagating modes. These modes are normalized with respect to the inner product (2.5) . One expansion for the scalar field ϕ is then

$$\phi = \int d\omega(a_\omega f_\omega + a^\dagger_\omega {f_\omega}^* + d_\omega j_\omega + d^\dagger_\omega {j_\omega}^*). \tag{3.25}$$

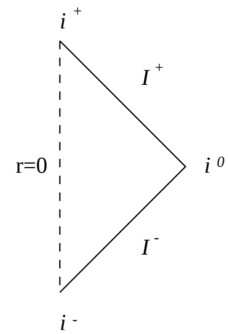

Figure 1: Penrose diagram for 3+1 dimensional Minkowski spacetime. The radial-time plane is shown, and each point is an S^2. The conventions are that $\mathcal{I}^-$ is past null infinity, $\mathcal{I}^+$ is future null infinity. i_- is past timelike infinity, and i_+ is future timelike infinity. i_o is spacelike infinity. In figures below wavy denote curvature singularities. The dashed line above is the origin of radial coordinates.

The mode operators a_ω and d_ω are taken to annhilate the global inertial vacuum $|0>$

$$a_\omega|0>= d_\omega|0>= 0 \tag{3.26}$$

for all $\omega > 0$. We will take $|0>$ to be the quantum state of the scalar field.

For the Rindler observer, we define the modes q_ω, ${q_\omega}^*$, p_ω and ${p_\omega}^*$ given by

$$q_\omega = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega v}, \qquad p_\omega = \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega u}, \tag{3.27}$$

which are only defined in the Rindler wedge. A second expansion for the scalar field ϕ is then

$$\phi = \int d\omega (b_\omega p_\omega + b_\omega^\dagger {p_\omega}^* + c_\omega q_\omega + c_\omega^\dagger {q_\omega}^*). \tag{3.28}$$

The Rindler mode operators $b_\omega^\dagger$ and $c_\omega^\dagger$ are creation operators for inward and outward propagating Rindler particles respectively. The number of particles that the accelerating observer measures near $\mathcal{I}^+$ is then given by (2.20), where the in-vacuum vacuum is defined with respect the the global inertial time coordinate, as in (3.26).

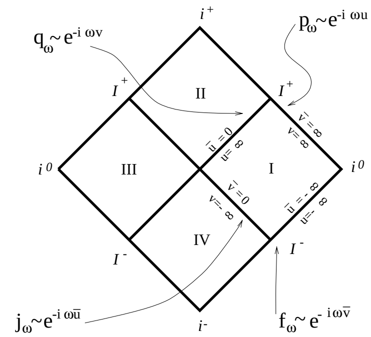

Figure 2: Penrose diagram for 1 + 1 dimensional Minkowski spacetime. Minkowski lightcone coordinates are $(\bar{u}, \bar{v})$. Region I is the wedge covered by the Rindler coordinates (u, v). There is a symmetrical wedge on the left hand side; the calculations below are done in region I. The modes which define positive frequency on each of the boundaries of region I are indicated.

Normalized wave packets

The mode functions are normalized in the sense of distributions, however each mode is not square integrable. To get a finite result for a particle production calculation, one needs to form square-integrable wave packets. Let $F_\omega(\bar{u}, \bar{v})$ be the solution to the wave equation which is equal to a specified positive frequency wave packet on $\mathcal{I}^-$,

$$F_\omega \to \int d\nu W(\nu - \omega) f_\nu(\bar{v}), \qquad \bar{v} \to \mathcal{I}^-. \tag{3.29}$$

Here $W(x)$ is a "window function" that is peaked about the origin and chosen such that the packet is peaked about $\bar{v}$ near $\mathcal{I}^-$. In particular, the function F_ω vanishes on $\mathcal{H}^+$. Rather than introducing corresponding new notation for all the modes, we will indicate the places in the calculations where it is necessary to sum up the plane wave modes to make normalizable states F, J, Q, P.

In the black hole calculation, boundary conditions on the scalar field ϕ are set on $\mathcal{I}^-$, so we will proceed analogously here. Given a positive frequency, outward propagating Rindler wave packet on $\mathcal{I}^+$, one wants to solve the wave equation to find the form of the wave packet in the far past. One then decomposes this into a sum over positive and negative frequency parts with respect to the Rindler coordiante v. The calculation is most simply carried out mode by mode, *i.e.* for $\phi \to e^{-i\omega u}$ on $\mathcal{I}^+$. In the Rindler wedge, the past boundary is $\mathcal{H}^-$ plus $\mathcal{I}^-$. Because spacetime is flat, there is no scattering of the scalar field ϕ. Therefore, the wave packet above propagates from $\mathcal{I}^+$ to $\mathcal{H}^-$, and none reaches $\mathcal{I}^-$. The particle production comes solely from the change of basis, *i.e.* from different definitions of time.

Particle Production

To compute the flux of Rindler particles across $\mathcal{I}^+$ we only need the Bogoliubov coefficients as in (2.17)

$$\alpha_{\omega\omega'} = (p_\omega, j_{\omega'})_{\mathcal{H}^-}, \qquad \beta_{\omega\omega'} = -i\alpha_{\omega,-\omega'}, \tag{3.30}$$

where the first integral is taken over the past Cauchy horizon $\mathcal{H}^-$. The mode functions satisfy $\partial_{\bar{u}} p_\omega{}^* = (i\omega/a\bar{u})p_\omega$, so that

$$\begin{aligned} \alpha_{\omega\omega'} &= \frac{-1}{4\pi\sqrt{\omega\omega'}}\int_{-\infty}^{0} d\bar{u}(\omega' - \frac{\omega}{a\bar{u}})e^{i\omega'\bar{u}}e^{i\frac{\omega}{a}ln(-\bar{u})} && (3.31)\\ &= \frac{i}{2\pi}\frac{1}{\sqrt{\omega'\omega}}(i\omega')^{-i\frac{\omega}{a}}\Gamma(1+i\frac{\omega}{a}), && (3.32)\end{aligned}$$

where $\Gamma(s) = \int_0^\infty e^{-z}z^{s-1}dz$, and we have used $\Gamma(1+s) = s\Gamma(s)$. This is the same expression that we will find for the Bogoliubov coefficients $\alpha_{\omega\omega'}$ in the black hole case, with the acceleration a being replaced by the surface gravity of the black hole. With a bit more analysis, which we defer until the black holes calculation, we will find after restoring factors of Planck's constant $\hbar$ the result for the number of particles produced in each Rindler mode

$$< N_\omega^{rind} >= \frac{1}{e^{\frac{2\pi\omega}{\hbar a}} - 1}, \tag{3.33}$$

which is a black body or thermal spectrum, with temperature

$$T = \hbar\frac{a}{2\pi}. \tag{3.34}$$

4 Black Holes

A stationary black hole spacetime[1] has a killing vector ξ^a which is normal to the horizon, and whose norm $\xi^a\xi_a = 0$ on the horizon. The surface gravity κ is defined by $\nabla^b(\xi^a\xi_a) = -2\kappa\xi^b$ on the horizon. The horizon area A is the area of the intersection of the horizon with a constant time slice, which is a two-sphere in all of the cases considered here.

According to Birkhoff's theorem, the Schwarzschild metric below is the unique spherically symmetric solution to the vacuum Einstein equation $R_{ab} = 0$,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V} + r^2 d\Omega^2, \qquad V(r) = 1 - \frac{2M}{r}. \tag{4.35}$$

Here $d\Omega^2$ is the volume element on the unit 2-sphere. The spacetime has a black hole horizon where the norm of the time-translation killing vector $\frac{\partial}{\partial t}$ vanishes. In the coordinates (4.35) the horizon is at $r = 2M$ and has area $A = 4\pi M^2$. The parameter M is the ADM mass of the spacetime. For any static black hole with metric of the form (4.35), possibly with a different function $V(r)$, the surface gravity is given by $\kappa = \frac{1}{2}V'(r_H)$, where r_H is the horizon radius. The metric (4.35) has a coordinate singularity at $r = 2M$ and a curvature singularity at $r = 0$.

For the particle production calculation, we will need the black hole metric in several different coordinate systems,

$$\begin{aligned} ds^2 &= V(r)(-dt^2 + dr^{*2}) + r^2 d\Omega^2 & (4.36)\\ &= -\frac{2M}{r}e^{-r/2M}e^{(v-u)/4M}dudv + r^2 d\Omega^2 & (4.37)\\ &= -\frac{2M^3}{r}e^{-r/2M}dUdV + r^2 d\Omega^2. & (4.38) \end{aligned}$$

The radial coordinate r^* is known as the tortoise coordinate, u and v are a pair of ingoing and outgoing null coordinates and, finally, U and V are ingoing and outgoing null Kruskal coordinates. The relations between the different coordinates are given by

$$dr^* = \frac{dr}{V(r)}, \quad r^* = r + 2Mln(\frac{r}{2M} - 1) \tag{4.39}$$

[1] This will not be a comprehensive introduction to black holes! See, *e.g.* [6] for details, proofs, and further properties.

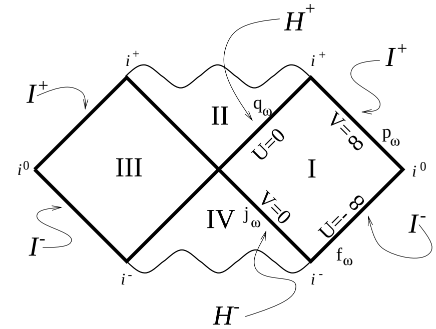

Figure 3: Penrose diagram for an eternal Black Hole (Extended Schwarzchild): Regions I and III are asymptotically flat, Region II is the black hole, and Region IV is the white hole. For an observer in Region I,$\mathcal{H}^+$ is the (future) black hole horizon and $\mathcal{H}^-$ is the past black hole, or white hole, horizon. (U, V) are the Kruskal coordinates.

$$u = t - r^*, \quad v = t + r^* \tag{4.40}$$

$$U = -e^{-u/4M}, \quad V = e^{v/4M}, \tag{4.41}$$

where factors of r are understood to be implicit functions of $r*$, u, v, or U, V respectively. The definition of dr^* is rather general, though usually one can't do the integral explicitly. The black hole horizon is regular in the Kruskal coordinates. One finds that the Schwarzschild coordinates (4.35) actually only cover part of the manifold, but that the Kruskal coordinates cover the extended spacetime. These features of the geometry are diplayed in the conformal (or Penrose) diagrams, figures (2) and (3).

Finally, we will look at solutions to the scalar wave equation in the Schwarzchild geometry. Writing ϕ as the product

$$separate\phi_{\omega lm}(t, r*, \Omega) = \psi(r*) Y_{lm}(\Omega) e^{-i\omega t}, \tag{4.42}$$

the wave equation (2.4) reduces to the radial equation

$$(\partial_t^2 - \partial_{r*}^2 + W(r))\psi = 0, \qquad W(r) = (1 - \frac{2M}{r})(\frac{2M}{r^3} + \frac{l(l+1)}{r^2}). \tag{4.43}$$

Note that in terms of the tortoise coordinate, the horizon $r = 2M$ is $r^* \to -\infty$, whereas in the asymptotically flat limit $r \to \infty$, we also have $r^* \to \infty$.

In the asymptotic region $r^* \to \infty$, the potential behaves as $W(r) \to \frac{l(l+1)}{r^2}$, and near the horizon $r* \to -\infty$, we have $W(r) \to e^{r*/2M}$. Therefore, both near infinity and near the horizon, the solutions $\phi_{\omega lm}$ are plane waves in $t \pm r*$, *i.e.* plane waves in u, v. These solutions to the wave equation will be used to define the bases for the Hilbert space of states. For notational simplicity, as in section (2), we will supress the l, m subscripts in the following calculations.

5 Particle Emission from Black Holes

Our strategy will be to first present Hawking's original calculation of particle production, which is done in a gravitational collapse spacetime. In the following sections, we will then compute the same result for the eternal black hole, as well as an analogous result for a charged eternal black hole in a spacetime which is asymptotically deSitter.

Defining the Problem

First let us outline the idea. Hawking [1] originally did the calculation of particle emission for a black hole that is formed by gravitational collapse. In the far past, the spacetime is nearly Minkowski, the largest gravitational effects being at the surface of the star, and we can assume that the quantum state is empty of *in*-particles near $\mathcal{I}^-$. We will call this state $|0>_{in}$. The star collapses to form a black hole. Hawking found that near $\mathcal{I}^+$, the state $|0>_{in}$ contains a thermal flux of *out*-particles. The particles produced are known as Hawking radiation.

There is no white hole horizon in the collapse spacetime, since $\mathcal{H}^-$ is replaced by the interior of the collapsing star. From the conformal diagram in figure (4), one sees that $\mathcal{I}^-$ is a Cauchy surface. In order to choose a set of basis functions that define particle states in the far past, one must choose a time coordinate with which to define positive frequency oscillations on $\mathcal{I}^-$. We will take the the early time positive frequency modes to be the solutions f_ω to the wave equation that behave near $\mathcal{I}^-$ like

$$f_\omega(u, v) \to e^{-i\omega v} \tag{5.44}$$

Far from the star spacetime becomes flat and v becomes an ingoing null coordinate for the flat space wave equations. Therefore, this choice of positive

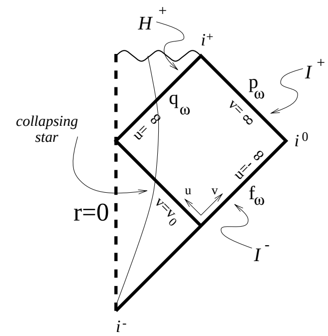

Figure 4: Penrose diagram for a black hole formed via gravitational collapse: The boundary of the collapsing star is shown. The star interior covers up regions III and IV of the extended black hole spacetime. Spacetime curvature is small inside the star. At some point during collapse, the star falls within its event horizon, and the black hole forms.

frequency modes corresponds to the usual Minkowski particle states. Note that v is the affine parameter for the null geodesic generators of $\mathcal{I}^-$.

Define creation and annihilation operators $a^\dagger_\omega$, a_ω for these, as in (2.7), via the expansion

$$\phi(u,v) = \int d\omega (a_\omega f_\omega + a^\dagger_\omega f^*_\omega) \tag{5.45}$$

The vacuum is then taken to satisfy

$$a_\omega |0>_{in} = 0, \tag{5.46}$$

for all $\omega > 0$. Note that this state is annhilated by the a_ω at all times. The label *in* refers to the fact that the boundary conditions on the modes f_ω are fixed on $\mathcal{I}^-$.

In order to define a complete set of particle states at late times, we must define modes on both $\mathcal{I}^+$ and $\mathcal{H}^+$, because $\mathcal{I}^+$ itself is not a Cauchy surface. On $\mathcal{I}^+$ we take the *out*-states to be solutions to the wave equation with boundary conditions that on $\mathcal{I}^+$

$$p_\omega \to e^{-i\omega u}. \tag{5.47}$$

The coordinate u is an outgoing null coordinate and is the affine parameter for the null geodesic generators of $\mathcal{I}^+$. Again, this choice of positive frequency late time modes coincides with the usual choice in Minkowski spacetime.

In order to form a complete basis, we must add modes which define particle states on $\mathcal{H}^+$ and it's extension through the collapsing matter. Here we cannot make a choice based on a flat spacetime limit. One approach to this problem is as follows [1]. Choose any set of modes q_ω that are well behaved on $\mathcal{H}^+$. The choice of quantum state $|0>_{in}$ implies that at early times, the density matrix[2] of the system is simply

$$\rho = |0>_{in\ in} < 0|, \tag{5.48}$$

the density matrix for the "pure state" $|0>_{in}$. The operator ρ can be expanded in either the *in* or *out* basis. Expanding ρ in the f_ω, q_ω basis, it is a product of the "$\mathcal{H}^+$" Fock space, constructed with the mode operators $c^\dagger_\omega$ and the "$\mathcal{I}^+$" Fock space, constructed with the operators $b^\dagger_\omega$. The expectation value of any operator O^{AF} that only depends on the degrees of freedom in the asymptotically flat region of the spacetime (region I in Fig 3) may be computed using the reduced density matrix $\rho^{red} \equiv Tr_{\{q\}}\, \rho$ as

$$< O^{AF} >= Tr(\rho^{red} O^{AF}) \tag{5.49}$$

The reduced density matrix, ρ^{red}, is the same for all bases related by unitary transformations to the chosen basis. Therefore $< O^{AF} >$ is independent of the choice of modes q_ω on the black hole horizon $\mathcal{H}^+$.

Therefore, the scalar field ϕ can also be expanded in the *out*-basis,

$$\phi = \int d\omega (b_\omega p_\omega + b^\dagger_\omega p_\omega * c_\omega q_\omega + c^\dagger_\omega q_\omega *) \tag{5.50}$$

As discussed in the Rindler example, one must use normalized wave packets to have a finite result for the number of particles produced in given a frequency interval, per unit time. Again, here we will do the calculation individually for each eigenmode and assemble wave packets at the end.

Of course, solving the wave equation in the black hole spacetime is harder than in the Minkowski case! In the black hole case, we don't know global

[2]The density matrix formulation of quantum mechanics is a generalization of the standard Schroedinger/Heisenberg wave mechanics, which is needed for quantum statistical mechanics.

analytic solutions. Consider a wave packet peaked about frequency ω that propagates inward from $\mathcal{I}^-$ towards the horizon of an eternal black hole. Roughly speaking, the wave scatters in two parts. A fraction $1-\Gamma_\omega$ of the packet backscatters off the curved geometry, *i.e.* due to the potential $W(r)$ in (4.43), and propagates out to $\mathcal{I}^+$, essentially without a change of frequency. The remaining fraction Γ_ω propagates parallel to $\mathcal{H}^-$ and is absorbed by the black hole horizon. It is this second portion that leads to the particle production. Therefore, we can write $f_\omega = {f_\omega}^{(1)} + {f_\omega}^{(2)}$, where the superscripts (1) and (2) denote these two parts, and similarly for the functions j_ω, p_ω and q_ω. We can also write for the Bogoliubov coefficients and the scattering coefficient Γ_ω,

$$\alpha_{\omega\omega'} = {\alpha_{\omega\omega'}}^{(1)}\delta_{\omega'\omega} + {\alpha_{\omega\omega'}}^{(2)}, \qquad \beta_{\omega\omega'} = {\beta_{\omega\omega'}}^{(2)} \tag{5.51}$$

$$\Gamma_\omega = \int d\omega'(|{\alpha_{\omega\omega'}}^{(2)}|^2 - |{\beta_{\omega\omega'}}^{(2)}|^2) \tag{5.52}$$

To compute particle production, one can ignore the backscattered (1) component of the wave. In addition, for simplicity we will drop the superscript (2) on the coefficients in the following.

Hawking did the calculation by studying a wave propagating backwards in time in the collapsing star spacetime. Choose boundary conditions such that the wave is positive frequency on $\mathcal{I}^+$, so that the scalar field $\phi \to p_\omega$ as in (5.47). The goal is then to solve for the behavior of the scalar field ϕ on $\mathcal{I}^-$ and to decompose the wave into positive and negative frequency parts there. Given this choice of boundary conditions, the wave propagates backwards in time, and the collapsing star geometry sets the natural definitions of positive and negative frequency in the far past.

In section (6), we will compute black hole radiation in the extended, eternal Schwarzschild spacetime with a particular choice of positive frequency on $\mathcal{H}^-$. This choice is dictated by the results in the collapsing black hole calculation. Importantly, learning what choice to make for positive frequency modes on horizons allows one to extend the calculation of Hawking radiation to black holes in spacetimes that are not asymptotically flat [5]. We will outine one such calculation in section (6).

The Calculation

1) The mode (5.47) propagates along a path γ that goes from $\mathcal{I}^+$ along a geodesic $u = u_1$ passing close to the black hole horizon $\mathcal{H}^+$. The ray passes

through the collapsing star and then propagates out to $\mathcal{I}^-$ along a geodesic $v = v_1$, which is close to $v = v_0$. The ray $v = v_0$, shown in figure (4), is the last inward propagating ray on the surface of the star that reaches $\mathcal{I}^+$. Inward propagating rays with $v > v_o$ enter the black hole. In the extended spacetime $v = v_o$ would be the white hole horizon $\mathcal{H}^-$.
2) The ray γ is connected to $\mathcal{H}^+$ and $v = v_o$ by a geodesic deviation vector ϵn^a with ϵ small and positive. On the part of the path that passes close to $\mathcal{H}^+$, n^a is tangent to a null geodesic which is ingoing at $\mathcal{H}^+$. The normalization is fixed by the condition $n^a l_a = -1$, where l^a is a null geodesic generator of $\mathcal{H}^+$.
3) Let p^a be tangent to an ingoing null geodesic at $\mathcal{H}^+$, $p^a = \frac{du}{d\lambda}\frac{\partial}{\partial u}$. Note that p^a is parallel to n^a and therefore satisfies

$$p^b = A^2 n^b. \tag{5.53}$$

Solving the geodesic equation for p^a near $\mathcal{H}^+$ gives the affine parameter λ in terms of the coordinate u,

$$\lambda = -B^2 e^{-\kappa u} = B^2 U \tag{5.54}$$

where κ is the surface gravity of the black hole, and U is the Kruskal coordinate defined in (4.39). This expression will be useful below. For the Schwarzchild case, $\kappa = 1/4M$, but the relation (5.54) will generalize to other cases as well. The affine parameter $\lambda = 0$ on $\mathcal{H}^+$.
4) The affine parameter is a good coordinate near $\mathcal{H}^+$, while u is not. So the deviation vector connects the two null geodesics, the horizon at $\lambda = 0$ and the ray γ at λ, where $\lambda < 0$.
5) In these local inertial coordinates, the geodesic equation is simply $\frac{dp^\mu}{d\lambda} = \frac{d^2x^\mu}{d\lambda^2} = 0$, so that

$$\lambda p^\mu = x^\mu(\lambda) - x^\mu(0) = -\epsilon n^\mu, \tag{5.55}$$

where the last equality follows from the definition of the deviation vector in point (2) above. Equations (5.55) and (5.53) then imply that

$$\epsilon = -\lambda A^2 \tag{5.56}$$

6) Next, we trace the ray γ through the collapsing star, and back to $\mathcal{I}^-$. In the conformal diagram, the ray bounces off the origin of coordinates and follows a null geodesic of constant $v < v_0$, which is near $v = v_0$. The two

geodesics are still connected by ϵn^a. Since spacetime is approximately flat on this part of γ, we have

$$v_0 - v = \epsilon = -\lambda A^2 = C^2 e^{-\kappa u}, \tag{5.57}$$

which holds[3] on $\mathcal{I}^-$.

Equation (5.57) is the desired relation between u and v. For a solution to the scalar wave equation $\nabla_a \nabla^a \phi = 0$ that has the boundary condition $\phi \sim e^{-i\omega u}$ at $\mathcal{I}^+$, we have on $\mathcal{I}^-$

$$\begin{aligned} \phi &\sim e^{i\frac{\omega}{\kappa} ln(\frac{v_0 - v}{C^2})}, \qquad v < v_0 & (5.58) \\ \phi &\sim 0, \qquad\qquad\qquad v > v_0. & (5.59) \end{aligned}$$

The wave vanishes for $v > v_0$ because it would have had to come out of the black hole horizon to reach this part of $\mathcal{I}^-$. Proceeding as before, one finds the expressions for the Bogoliubov coefficients

$$\begin{aligned} \alpha_{\omega\omega'} &= (p_\omega, f_{\omega'})_{\mathcal{I}^-} = \frac{1}{2\pi\sqrt{\omega\omega'}} \int_{-\infty}^{0} dv \left(\omega' - \frac{\omega}{\kappa v}\right) e^{i\omega' v} e^{i\frac{\omega}{\kappa} ln(-v)} & (5.60) \\ &= \frac{1}{i\pi\sqrt{\omega\omega'}} (i\omega')^{-i\frac{\omega}{\kappa}} \Gamma(1 + i\frac{\omega}{\kappa}) & (5.61) \\ \beta_{\omega\omega'} &= -i\alpha_{\omega,-\omega'}, & (5.62) \end{aligned}$$

where we have set $v_0 = 0$ in the above expressions.

The Bogoliubov coefficient $\alpha_{\omega\omega'}$ is analytic in the lower half of the complex ω' plane, because it is the fourier transform of a function which vanishes for $v > 0$. The coefficient $\alpha_{\omega\omega'}$ has a logarithmic branch point at $\omega' = 0$, so the branch cut extends into the upper half plane. Therefore, we have

$$|\alpha_{\omega\omega'}| = e^{\pi\omega/\kappa} |\beta_{\omega\omega'}|. \tag{5.63}$$

The spectrum of produced particles that then follows, making use of (2.20) and (5.51), is given by

$$< N_\omega^{bh} > \; = \int d\omega' |\beta_{\omega\omega'}|^2 \tag{5.64}$$

[3] On the conformal diagram, the deviation vector appears to flip direction when it "turns the corner". This is simply because the vertical left hand boundary is the origin of coordinates, so that the rays are reflected rather than continued. Note that the signs are consistent through the chain of equalities in (5.57). I would like to thank the students in my 1999 General Relativity class for patiently helping to sort out the signs.

$$= \frac{\Gamma_\omega}{e^{\frac{2\pi\omega}{\hbar\kappa}} - 1} \tag{5.65}$$

This is a black body or thermal spectrum, with temperature

$$T = \hbar\frac{\kappa}{2\pi}, \tag{5.66}$$

with $\kappa = 1/4\pi$ for Schwarzschild.

The coefficient Γ entered in our discussion of normalized wave packets, as the portion of the wave which propagates close to the horizon, through the colllapsing star, and back out to $\mathcal{I}^-$. This is almost identical to the fraction which would propagate into the white hole horizon $\mathcal{H}^-$ if we were working in the extended spacetime, rather than the gravitational collapse case. But this in turn is equal to the fraction of a wave which is absorbed by the black hole horizon $\mathcal{H}^+$ for a wave which starts at $\mathcal{I}^-$. So Γ_ω is just the classical absorption coefficient for scattering a classical scalar field off a black hole. Direct calculation gives

$$\Gamma_\omega \to 1,\ \omega M \gg 1, \quad \Gamma_\omega \to \frac{A}{4\pi}\omega^2,\ \omega M \ll 1. \tag{5.67}$$

The large energy limit is just the particle limit, in which everything is absorped.

One fascinating implication is that the classical black hole mechanics theorems and the laws of thermodynamics have more than a formal analogy. A black hole radiates with temperature $T = \hbar\frac{\kappa}{2\pi}$, and has an entropy

$$S_{bh} = \frac{1}{4}A! \tag{5.68}$$

Generality and Back Reaction

Hawking also calculated particle production in quantum fields by charged and rotating black holes. Calculations have also been done for emission of fermions and gravitons, linearized perturbations of the metric. In all of these cases one finds a thermal spectrum,

$$< N^{bh}_\omega >= \frac{\Gamma_\omega}{e^{\frac{2\pi(\omega-\mu)}{\hbar\kappa}} \pm 1}, \tag{5.69}$$

where the +1 corresponds to fermions and -1 to bosons. In thermodynamics, μ is called a chemical potential. For black hole emission, μ is such that a charged black hole preferentially emits charged, massless particles of the same sign as its own charge. Rotating black holes preferentially emit particles with the same sense of angular momentum. Hence black holes can spindown via Hawking radiation and also discharge, if there are fields which carry the same kind of charge as the black hole. Another generalization of interest is to black branes in higher dimensions, which are important in string theory and will be discussed briefly below in section (7).

In the preceeding calculation, the spacetime metric was fixed. Even though we don't have a quantum theory of gravity to determine how the metric evolves with the quantum particle emission, it is assumed that the mass of the black hole decreases. For a neutral black hole, the temperature increases as the mass decreases, so the rate of black hole evaporation increases with time. Very small black holes have very large curvatures, and at some point the classical gravity description is not valid. So the endpoint of this run away evaporation is not something we can compute and has been the subject of much debate.

However, the situation is rather different for particle production from charged black holes. A static, spherically symmetric, charged black is a solution to the Einstein-Maxwell equations, *i.e.* equation (1.1) with T_{ab} given by the stress-energy of the Maxwell field. We will take the black hole charge Q to be positive. The Reissner-Nordstrom spacetime for an electrically charged black hole is given by

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V} + r^2 d\Omega^2, \quad V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \tag{5.70}$$

$$A_b dx^b = -\frac{Q}{r} dt. \tag{5.71}$$

Here A_b is the $U(1)$ electromagnetic gauge potential. The spacetime (5.70) describes a black hole, *i.e.* there is a horizon, when $M \geq Q$. For $M < Q$ there is no horizon and the spacetime has a naked singularity. The case $M = Q$ is called an extremal black hole.

For the Reissner-Nordstrom black holes, the temperature is still given by $T = \hbar \frac{\kappa}{2\pi}$ with the surface gravity κ calculated from the metric (5.70). For $M \gg Q$, the temperature reduces to the Schwarzchild result. However, as $M \to Q$ the surface gravity $\kappa \to 0$, with $\kappa = 0$ for $M = Q$. Therefore, the temperature vanishes for an extremal black hole. So, for charged black holes, if we assume that there are no charged fields present to discharge the hole, then the semiclassical calculation says that a black hole with $M > Q$ evaporates down to $M = Q$, at which point the evaporation stops. We will return to this picture in connection with the positive mass theorems for black holes, and quantum mechanical ground states in string theory in section (7).

6 Extended Schwarzchild and Reissner-Nordstrom deSitter Spacetimes

Extended Schwarzchild

In the extended Schwarzchild spacetime, also known as the eternal black hole, shown in figure (3), one basis consists of the modes $\{f_\omega, j_\omega\}$ with boundary conditions specified on $\mathcal{I}^-$ and $\mathcal{H}^-$ respectively. A second basis consists of the modes $\{p_\omega, q_\omega\}$ with boundary conditions specified on $\mathcal{I}^+$ and $\mathcal{H}^+$ respectively. On $\mathcal{I}^-$ and $\mathcal{I}^+$ we choose the same modes as before, (5.44) and (5.47).

On the black hole and white hole horizons, we will define positive frequency modes so that the resulting particle production is the same as in the collapse spacetime. Indeed, equation (5.54) implies that the correct choice on the horizons is to use the null Kruskal coordinates (U, V) defined in (4.39). Note that the coordinates U, V are affine parameters for the null geodesic generators of the horizons, so this choice is consistent with the choices of (u, v) at null infinity. We then have

$$j_\omega \rightarrow \frac{1}{\sqrt{2\omega}} e^{-i\omega U}, \quad \text{near } \mathcal{H}^- \tag{6.72}$$

$$q_\omega \rightarrow \frac{1}{\sqrt{2\omega}} e^{-i\omega V}, \quad \text{near } \mathcal{H}^+ \tag{6.73}$$

To find the Bogoliubov coefficients $\alpha_{\omega\omega'}$, the computation in (5.60) is replaced by an integral over $\mathcal{H}^-$, as was done for the Rindler spacetime in (3.30). The integral is then the same as in equations (3.31) and (5.60), and the thermal spectrum follows as before.

Charged Black Holes in DeSitter

A deSitter spacetime is a spacetime of constant positive scalar curvature, and is a solution to the Einstein equation with cosmological constant $\Lambda > 0$, *i.e.* $G_{ab} = 8\pi\Lambda g_{ab}$. A particular slicing of deSitter describes the Inflationary Universe. A Reissner-Nordstrom-deSitter, or RNdS, spacetime describes an eternal charged black hole in a spacetime which is asymptotically deSitter, rather than asymptotically flat. The metric and gauge field are given by the expressions in (5.70), but with the radial function $V(r)$ given by

$$V(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{1}{3}\Lambda^2 r^2. \tag{6.74}$$

For a range of values of Q and M, the spacetime has three Killing horizons; inner and outer black hole horizons and a Cauchy horizon, called the deSitter horizon. This implies that there are two sources of particle production in an RNdS spacetime, the black hole horizon and the deSitter horizon [9]. One interesting question that we will address below is whether these two sources can ever be in a state of thermal equilibrium [5].

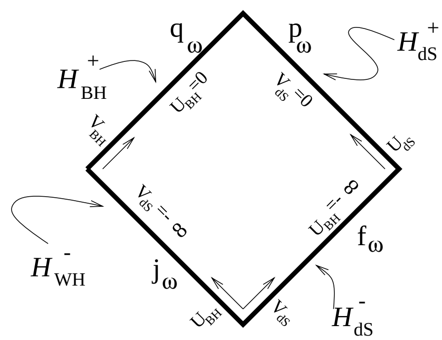

Figure 5: A part of the conformal diagram for RNdS (a charged black hole in asymptotically deSitter spacetime). The black hole, white hole, past and future deSitter horizons are indicated.

The conformal diagram for the relevant portion of RNdS is shown in figure (5). The region is bounded by the white hole, black hole, past and future deSitter horizons. Following the discussion for the extended Schwarzschild spacetime, we define positive frequency on each of these horizons by a Kruskal-type coordinate, *i.e.* a coordinate which is an affine parameter for the null geodesic generators of that horizon. Explicity, letting $u = t + r^*$ and $v = t - r^*$, the Kruskal coordinates are given by

$$U_{bh} = -\frac{1}{\kappa_{bh}} e^{-\kappa_{bh} u}, \qquad V_{bh} = \frac{1}{\kappa_{bh}} e^{\kappa_{bh} v} \tag{6.75}$$

$$U_{ds} = \frac{1}{\kappa_{ds}} e^{\kappa_{ds} u}, \qquad V_{dS} = -\frac{1}{\kappa_{ds}} e^{-\kappa_{ds} v} \tag{6.76}$$

Near the black hole horizon, the metric is then well behaved and has the limitting form

$$ds^2 \approx \kappa_{bh} dU_{bh} dV_{bh}, \tag{6.77}$$

Similarly, one can show that the coordinates (U_{ds}, V_{ds}) are also good near the deSitter horizon.

The Klein-Gordon equation for ϕ near any of the horizons reduces to the free wave equation. As in the case $\Lambda = 0$, the potential $W(r)$ due to the background gravitational field decays exponentially near a horizon. Consider then a pure positive frequency, outgoing wave near the deSitter horizon at late time, $p_\omega \sim e^{-i\omega U_{ds}}$. In the geometrics optics limit, finding the form of this wave propagated back to the white hole horizon reduces to finding the dependence of the coordinate U_{ds} on the coordinate U_{bh}. Using the expressions in (6.75) , it follows that on the white hole horizon the quantity $G_\omega(U_{bh}) \equiv p_\omega(U_{ds}(U_{bh}))$ behaves as

$$G_\omega(U_{bh}) \sim e^{-i\omega\xi^2(\frac{-1}{U_{bh}})^\eta}, \qquad U_{bh} < 0 \tag{6.78}$$

$$G_\omega(U_{bh}) \sim 0, \qquad U_{bh} > 0. \tag{6.79}$$

where $\eta \equiv \kappa_{ds}/\kappa_{bh}$ and $\xi^2 \equiv \frac{1}{\kappa_{ds}}(\frac{1}{\kappa_{bh}})^\eta$. The Bogoliubov coefficients are then given by

$$\beta^{bh}_{\omega\omega'} = \frac{1}{\sqrt{2\pi\omega}} \int dU_{bh} e^{-i\omega' U_{bh}} G_\omega(U_{bh}). \tag{6.80}$$

Similiarly, there is emission "from" the deSitter horizon as seen by an observer outside the black hole horizon at late times. Consider a positive frequency wave which is entering the black hole horizon, $q_\omega \sim e^{-i\omega V_{bh}}$. In the geometrics optics approximation on the past deSitter horizon, the quantity $F_\omega(V_{dS}) \equiv q_\omega(V_{bh}(V_{dS})$ is given by

$$F_\omega(V_{dS}) \sim \frac{1}{\sqrt{2\pi\omega}} e^{-i\omega\mu^2(\frac{-1}{V_{dS}})^{\frac{1}{\eta}}}, \qquad V_{dS} < 0 \tag{6.81}$$

$$F_\omega(V_{dS}) \sim 0, \qquad V_{dS} > 0, \tag{6.82}$$

where $\mu^2 = 1/\kappa_{bh}(1/\kappa_{ds})^{\frac{1}{\eta}}$. Similarly to equation (6.80), the Bogoliubov coefficients $\beta^{ds}_{\omega\omega'}$ are given in terms of the fourier transform of (6.81). For general values of Q and M, the functions F_ω and G_ω appearing in (6.78) and (6.81) are related according to

$$G_\omega(x) = F_{\frac{\omega}{\eta^2}}(x^{\eta^2}). \tag{6.83}$$

We see that the two functions are equal for $\eta = 1$, which occurs when $|Q| = M$. Therefore $\beta^{bh}_{\omega\omega'} = \beta^{ds}_{\omega\omega'}$ if and only if $|Q| = M$. This implies that for each horizon, the flux of particles absorbed is equal to the flux of particles emited.

The spectrum of emitted particles is given by $N_\omega = \int d\omega' |\beta_{\omega\omega'}|^2$. We can estimate the above integrals using the stationary phase approximation. It is simpler to work with the coefficients $\alpha_{\omega\omega'} = -i\beta_{\omega,-\omega'}$. For the case $|Q| = M$, when the surface gravities or temperatures are equal, we have

$$\begin{aligned} \alpha_{\omega\omega'} &= \frac{-1}{2\pi\sqrt{\omega\omega'}} \int_{-\infty}^{0} dU_{bh} (\omega' + \frac{\omega}{\kappa^2 {U_{bh}}^2}) e^{i\omega' U_{bh}} e^{i\frac{\omega}{\kappa^2 U_{bh}}} && (6.84) \\ &= \frac{-1}{2\pi\kappa} \frac{\omega'}{\omega} \int_{-\infty}^{0} dz (1 + \frac{1}{z^2}) e^{i(z+1/z)\sqrt{\omega\omega'}/\kappa}. && (6.85) \end{aligned}$$

For large ω' the stationary phase approximation gives

$$\alpha_{\omega\omega'} \approx \frac{-1}{2\pi\kappa} \frac{\omega'}{\omega} e^{-\frac{2i}{\kappa}\sqrt{\omega\omega'}}. \tag{6.86}$$

As before, the Bogoliubov coefficients $\beta_{\omega\omega'}$ are obtained by analytically continuation. Noting that (6.78) implies that $\alpha_{\omega\omega'}$ is analytic in the lower half ω' plane, we have

$$|\alpha_{\omega\omega'}|^2 = |\beta_{\omega\omega'}|^2 e^{\frac{4}{\kappa}\sqrt{\omega\omega'}} \tag{6.87}$$

Then (2.18) implies

$$\beta_{\omega\nu}\beta^*_{\omega'\nu} = \frac{e^{-i\nu(\omega-\omega')}}{e^{\frac{2}{\kappa}(\sqrt{\omega\nu}+\sqrt{\omega'\nu})} - 1}, \tag{6.88}$$

and finally we obtain the spectrum

$$\begin{aligned} < N_\omega > &= \int d\nu \beta_{\omega\nu}\beta^*_{\omega\nu} = \int_c^\infty d\nu (e^{\frac{4}{\kappa}\sqrt{\omega\nu}} - 1)^{-1} && (6.89) \\ &= \frac{\pi^2}{6}(\frac{\kappa^2}{8\omega} + \frac{\kappa}{2}\sqrt{\frac{c}{\omega}}) e^{-\frac{4}{\kappa}\sqrt{c\omega})}. && (6.90) \end{aligned}$$

The form of the spectrum depends on the infrared cutoff of the range of integration over frequencies. The integral converges if $c = 0$. However, one would expect that only wavelengths that are less than, or of order the deSitter horizon scale should be included, *i.e.* $c \approx A_{dS}^{-1/2}$. Note that the spectrum then is not a thermal black body spectrum, though the system is still in an equilibrium state.

There are several limits one can take in order to check this result. Letting $\kappa \to 0$ above, corresponds to keeping $|Q| = M$ and letting the cosmological constant Λ approach zero, so that the spacetime approaches extremal Reissner-Nordstrom. In this limit N_ω goes to zero, as it should. Secondly, one can set $Q = 0$, and then let $\Lambda \to 0$, so that the metric approaches Schwarzschild. The particle production (6.80) from the black hole can again be evaluated in the stationary phase approximation. One finds that the coefficients $\alpha_{\omega\omega'}$ approach those for a Schwarzschild black hole in this limit.

7 Black Hole Evaporation and Positive Mass Theorems

We close by pointing out a connection, between classical positive mass theorems in general relativity and lowest energy states in string theory, which is made via Hawking evaporation. Motivated by supergravity, spinor constructions have been used to prove that for asymptotically flat solutions to the Einstein-Maxwell equations, the ADM mass is always greater than or equal to the charge of the spacetime, $M \geq |Q|$. We refer the reader to the various papers for the full statement of the results, *e.g.* [11, 13] for the case without charge or horizons and [14, 15] for derivations which include charge and horizons. There are also a large number of subsequent results that include other gauge fields, different asymptotics, or higher dimensions, see *e.g.* [12, 16].

The bound is saturated, *i.e.* $M = |Q|$, if and only if the spacetime has a super-covariantly constant spinor. The super-covariant derivative operator[4] is given by the standard covariant derivative operator plus terms which depend on the gauge field strength. Such lowest mass spacetimes are called

[4]These derivative operators arise from supersymmetry transformations that leave the supergravity action invariant. However, one can view the resulting theorems as statements about the bosonic spacetimes, and the spinor field is used as a device.

Bogomolnyi-Prasad-Sommerfield (BPS) spacetimes, in analogy with magnetic monopoles that saturate a similar mass bound [22, 23] that also follows from a supersymmetric construction [24]. These spacetimes have the lowest mass for a fixed value of the charge. For spacetimes which are asymptotically flat, $|Q| = M$ occurs for extremal black holes, or higher dimensional extremal black branes that also have zero temperature. The non-extremal Riessner-Nordstrom black holes are a nice example of a known family of solutions, all more massive than the BPS state $M = Q$. After "turning on" quantum mechanics, one expects that the higher mass states will evaporate to the ground state by emission of quantum particles and that therefore BPS spacetimes are quantum mechanically stable.

Briefly,let's see what this looks like in string theory. In string theory, two desciptions of states arise. In perturbative string theory there is a Fock space of states for a 1+1 dimensional superconformal field theory. This is defined on the 1+1 dimensional world sheet of the string. The string propagates in $(9 + 1)$ dimensional Minkowski spacetime, or other allowed fixed spacetime geometries. In addition, perturbative string theory contains Dirichlet-branes, surfaces on which open strings can end. D-branes carry a variety of charges. The states are indexed by mass, spins, and charges. In another limit of string theory, one uses a supergravity field theory description. One thinks of a "state" as a spacetime with metric and gauge fields, indexed by mass, angular momenta, and charges. There is no well defined Hilbert space of quantum states in this regime. However, for BPS perturbative string states, there are many explicit calculations which display spacetimes that do have the matching quantum numbers. D-branes from the perturbative calculations show up in the supergravity spacetime solutions as black-branes which carry the right kinds of charges.

These BPS spacetimes are the lowest mass states in the positive mass theorems. In the context of the supergravity end of string theory, "excited spacetimes" decay to the lowest mass, zero temperature configurations by black hole evaporation. This is certainly an interesting picture. However there are untidy pieces that need explanation. For example, the family of charged "black holes" in Anti-deSitter (a spacetime of constant negative curvature) are given by (6.74) with $\Lambda < 0$. Anti-deSitter spacetimes are supergravity solutions with maximal supersymmetry. The lowest mass member, which does have a super-covariantly constant spinor, is not a black hole but a naked singularity. A naked singularity spells trouble in classical general relativity, and it is not clear how to think of this object in string theory.

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Finally, we remind the reader about the puzzle of what are the quantum mechanical microstates of black holes that are responsible for their thermodynamic attributes - temperature and entropy? A wide range of models have been studied. We will mention some of the string theory work. Other appraches include Euclidean quantum gravity [9] and the entropy of entanglement [19]. Certain D-brane plus attached open strings, which are BPS configurations that occur in perturbative string theory, are identified with certain types of extremal black holes. The statistical entropy of the D-branes plus strings configuration can be computed by explicitly counting states. In all cases where calculations have been done, begining with the work of [17], this has agreed with the black hole entropy $A/4$. In the string picture, Hawking evaporation is modeled as the emission of closed strings from slightly excited D-branes. Much work has been done on computing the low energy excitations of perturbative D-brane/string BPS configurations. These calculations have been compared to the presumed corresponding spacetime black hole evaporation calculations. Many of the calculations have agreed, and some have disagreed. The role of the horizon in defining the black hole entropy is still a mystery in the string calculations. There is no horizon since these are in flat spacetime. It is also not clear if the perturbative microstates are in any sense the same as microstates of the black hole. Nonetheless, the calculations are very interesting and understanding black hole thermodynamics continues to be an area of much current work.

However, when pursuing the definition and attributes of quantum gravity, it is perhaps well to remember that later understandings often "...formed just such a contrast with [one's] early opinion on the subject, ...as time is forever producing between the plans and decisions of mortals, for their own instruction, and their neighbor's entertainment." [5]

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[5] Jane Austen, "Mansfield Park".

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